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INVITED PAPER

Husserl on Geometry and Spatial Representation

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Abstract Husserl left many unpublished drafts explaining (or trying to) his views on spatial representation and geometry, such as, particularly, those collected in the second part of *Studien zur Arithmetik und Geometrie* (Hua XXI), but no completely articulate work on the subject. In this paper, I put forward an interpretation of what those views might have been. Husserl, I claim, distinguished among different conceptions of space, the space of perception (constituted from sensorial data by intentionally motivated psychic functions), that of physical geometry (or idealized perceptual space), the space of the mathematical science of physical nature (in which science, not only raw perception has a word) and the abstract spaces of mathematics (free creations of the mathematical mind), each of them with its peculiar geometrical structure. Perceptual space is proto-Euclidean and the space of physical geometry Euclidean, but mathematical physics, Husserl allowed, may find it convenient to represent physical space with a non-Euclidean structure. Mathematical spaces, on their turn, can be endowed, he thinks, with any geometry mathematicians may find interesting. Many other related questions are addressed here, in particular those concerning the a priori or a posteriori character of the many geometric features of perceptual space (bearing in mind that there are at least two different notions of a priori in Husserl, which we may call the conceptual and the transcendental a priori). I conclude with an overview of Weyl's ideas on the matter, since his philosophical conceptions are often traceable back to his former master, Husserl.

This paper is dedicated to my friend Claire Ortiz Hill for her sixtieth birthday.

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1 Introduction

What the subject matter of geometry is has not always been a matter of consensus, not even for the Greeks. Whereas Plato believed that geometry dealt with things that are not of this world, Aristotle thought it had only to do with abstract aspects of ordinary physical space and spatial objects. Both, however, never doubted geometry was in the business of searching for the truth and that it could tell us something about physical space.

Things changed as history unfolded. The intuitive acceptability of the foundations of classical geometry, successfully axiomatized by Euclid, has since antiquity been taken with a grain of salt (for Proclus, the 5th century commentator of Euclid, the non-existence of asymptotic straight lines is not intuitively obvious). For many the fifth postulate of the Euclidean system seemed, and for good reasons, to be lacking in intuitiveness. It took, however, more than two thousand years for us to realize that the system of ancient geometry, which in Euclid's axiomatic presentation is known as Euclidean geometry, is not the only possible consistent theory of space or even the only to have a rightful claim to being the *true* theory of space.

Throughout the nineteenth and into the twentieth centuries the true geometrical structure of physical space remained a much-debated scientific and philosophical question. The issue was further complicated by the fact, not always clearly recognized, that there are different ways of accessing spatial structure, and then different representations of physical space; there is the primitive, unreflective, operative representation, based on sense perception, but also geometric representation proper, based on geometrical intuition, a refinement of perceptual intuition. Taking these distinctions into consideration, some questions immediately arise; to mention a few: do the perceptual and geometric representations of space have the same features? How does geometrical intuition relate to perceptual intuition? What do these representations of space, the "naïve" and the scientific, have to do with space as it really is (if there is such a thing)?

In the beginning of his philosophical career Husserl was much involved with these and other questions concerning mathematical matters. Testimonies of his interests are the many drafts he left, written basically during the last decade of the Nineteenth century, in which a "philosophy of space" is sketched. And since Husserl almost never let a subject rest for good that once interested him, a phenomenological analysis of the intentional genesis of geometry and geometrical space appears also, and prominently, in his last published work, *The Crisis of European Science and Transcendental Phenomenology* (henceforth *Crisis*). It is worth noticing that both approaches bear substantial similarity; his ideas on the matter didn't change much in the span of a philosophical life. A work that must also be taken into consideration, although its main focus is not on the constitution of space, but of the spatial (rigid) body, is Husserl's *Thing and Space*: *Lectures of 1907* (*Hua* XVI, translated and edited by Richard Rodcewicz and published by Kluwer in 1997).

Unfortunately Husserl did not always express his ideas on geometry with the utmost clarity, so much so that scholars often disagree on how to interpret them. Much of the blame should be credited to the sketchiness of Husserl's notes, which were not written to be published, at least not in the format they reached us. However, if not the answers, at least the questions he raised were always clearly stated; here are some: does our intuitive representation of physical space contain a priori elements, in what sense? Is it at least partially based on conventions or hypotheses about physical reality? Is the geometry of physical space an empirical science, to what extent? How the scientific and pre-scientific representations of space relate to one another and to transcendent space? Is physical geometry a conceptual or an intuitive science? What perceptual and geometrical intuitions have to do with each other; can the former have any relevance for the latter? Does the mathematical conception of space derive from the intuitive representation of space, how? What role symbols and symbolic reasoning play in geometry, if any?

Although a few of the texts in which Husserl dealt with these questions were published, most, at least as they came to us, were not meant for the public eye. *Crisis*, in particular its appendix "The Origin of Geometry" (which contains an analysis of the intentional "genesis" of geometry and geometrical space) and *Logical Investigations* (where the nature of mathematics in general and pure geometry in particular are discussed) belong to the first group; the second part of *Studien zur Arithmetik und Geometrie* (Hua XXI), containing the bulk of Husserl's account of our representation of space from three perspectives, psychological, logical, and metaphysical, to the second. There are also relevant passing remarks here and there in other works; but the truth is that Husserl never produced a study on geometry nearly as complete as *Philosophy of Arithmetic* is for arithmetic. His views on geometry demand interpretation. My goal here is to articulate one.

2 The Origins

In the hands of its first practitioners, the Egyptians most notably, geometry was basically a technology for measuring lengths, areas and volumes. With the Greeks, it became a science; later axiomatized by Euclid of Alexandria (one of the persons of the sacred trinity of classical geometry with Apollonius and Archimedes). Euclid probably saw his axioms as statements concerning *constructions* that could in principle be carried out in ideal space with ideal straightedges and compasses, given the constructions we know we can in principle carry out in physical space, with real straightedges and compasses. Euclid was obviously able to appreciate the difference between ideal and real spaces and ideal and real constructions, but he was nevertheless convinced that our intuitions about real space and real constructions are somehow relevant to what we know about ideal space and ideal constructions.

All the five postulates of Euclid's tell us that something can be done: (1) we can draw a (straight) line connecting any two given points; (2) we can extend any given line; (3) we can draw a circle with center at any given point through any other given point; (4) we can, by moving them in space, if necessary, make any two given right angles coincide (i.e. all right angles are equal; (5) if two given lines are intercepted by a third so that the interior angles at one side are less than two right angles, we can then extend the given lines on this side till they meet.

The problem with the fifth postulate is obvious; whereas the remaining four involve finite tasks, the fifth doesn't. From the first to the fourth the things to be done are clearly delimited; the line of the first postulate begins and ends in points that are *given*, the extension mentioned in the second is limited by the case at hand—and so on. But we do not know a priori for *how long* we must extend the lines mentioned in the fifth before they meet. What if they don't? We would never know it by simply extending the lines indefinitely; and if not so, how? No one before Gauss in the last years of the eighteenth century seems to have thought the fifth postulate could actually be *false*; only that it lacked intuitive support and therefore should be proved.

But how to prove, on the sole basis of finite searches, that a possibly infinite search (for the point of intersection) comes to an end? The task was of course doomed to failure; Euclid's fifth postulate was eventually shown to be independent of the others. The search for a proof of the problematic postulate, however, lead, first Gauss and then Lobachevski and Bolyai, to the creation of the first non-Euclidean geometry, the now-called hyperbolic geometry, in which the *denial* of the fifth Euclidean postulate holds. That the given lines do *not* necessarily intercept implies that there is *more* than one parallel to a given line passing through a given point (the so-called Playfair axiom—there is *only one* straight line parallel to a given line passing through a given point—is logically equivalent to Euclid's fifth postulate). The case in which *no* such line exists is actually forbidden by the remaining postulates (otherwise there should be a line through the exterior point meeting the given line in *two* distinct points, which is impossible). But with convenient alterations in the system Riemann conceived the now-called elliptic geometry, in which no parallel exists to a given line through a given point.

By taking these new geometries seriously (which was not always the case, some Kantian-minded mathematicians insisted in considering Euclidean geometry the only *true* geometry and the others as more or less sterile formal exercises), mathematicians altered substantially the traditional comprehension of the nature of mathematics, no longer the study of (formal) aspects of given domains (physical space in the case of geometry) *only*, but purely formal theories, with no determinate domains, too; and not only for their own sake, but for scientific purposes as well, to *use* them in science and mathematics (which poses, of course, a problem: how can formal *inventions* be *scientifically* useful? But this is an altogether different problem, and here is not the place to deal with it).

Once new geometries were conceivable, it was difficult to avoid the question: is Euclidean geometry *really* the true geometry of physical space; or is it only hypothetically or conditionally true? Riemann and Helmholtz thought the validity of either Euclidean or non-Euclidean geometry for physical space rested ultimately on empirical facts or hypotheses. We live in a world, they say, in which some bodies, the so-called *rigid* bodies, *seem* able to move freely in space without changing size or shape (without some *physical* reason for so doing, that is), and the *simplest* (but not the only) space in which rigid bodies can so move is the Euclidean space (if we presuppose that space is actually *infinite*, not only *unbounded*—a difference our perception cannot appreciate—, both Euclidean and hyperbolic geometries satisfy the principle of free mobility. If we drop infiniteness, all the three non-Euclidean geometries qualify, for free mobility requires only that space be of constant curvature, which Euclidean, hyperbolic and elliptic spaces are—respectively with zero, negative and positive values). But how can we tell rigid bodies *really* exist; that there really are bodies that can move freely in space without deforming themselves?

For Helmholtz, we can tell this on a priori grounds (this is a typical transcendental argument). Since, he thinks, the rigidity of measuring rods is required by the very *notion* of measurement, the *possibility* of metric geometry requires the constancy of space curvature. Therefore, space has constant curvature out of necessity; its particular value, positive, negative or zero, however, must, he thinks, be empirically decided, by astronomical observations and measurements, for example[1] (or hypothesized, if no conclusive verification is or can be effected). Poincaré held similar views on the a priori character of the constancy of space curvature, but he believed no experiment would be conclusive for any particular value of this curvature, for the interpretation of experiments depends on physical theories we have no reason, logical, methodological, epistemological, or metaphysical, to prefer to the detriment of any geometry. For the sake of simplicity, he thinks, we can always endow space with zero curvature (and hence, Euclidean structure) and change physics accordingly. This view is usually labeled *conventionalism*. Notice, however, that the representation of space to which Poincaré is alluding relies on means of accessing spatial structure that goes beyond the purely intuitive.

Husserl, as we will see, thinks that both constancy of curvature and its particular value, positive, negative or zero, are *empirical* facts, only experience can—and *will*—tell us; not scientifically informed experiences though, but relevant sensorial perception pure and simple (and this is why a decision is necessarily reached, by default if needed). Husserl seems to reason thus: suppose the curvature of space is not, *as a matter of transcendent fact*, constant, but that we remain ignorant of it for all dimensions change in displacement in such a lawful manner that an *appearance* of constancy remains. Bodies not actually rigid will be, *ex hypothesi*, experienced as rigid. Wouldn't we then be *intuitively justified* in believing space had a constant curvature even if superhuman beings observing us from outside *our* world would tell a different story (because they can see the deformations we don't)? Suppose that, like the beings inside Poincaré's ball, that mistake a gradient of temperature that changes lengths in some lawful manner for a deformation of space, and adopt consequently a non-Euclidean geometry, our perception of space is influenced by physical factors we are unaware of. In this case we may be wrong as to the transcendent reality of our perceptual representation of space, but, nonetheless,

[1] Empirical verifications, in this case, involve not only direct perception via the outer senses (vision, tact, etc.), but mathematically informed experiments if necessary.

we are intuitively *justified* in believing space to be how we represent it to be. For Husserl, the *intuitive* representation of physical space cannot depend on *mediate* verifications regarding matters of actual fact. The *scientific* representation of space, of course, is an altogether different matter. Husserl believed we *naturally* and *irrecusably* represent space intuitively in a way that it is best described within reasonable limits of approximation by Euclidean geometry. What remains to be explained is how we do this.

Husserl came to philosophy by the way of mathematics at a time when the question about the geometrical structure of physical space was pressing for a solution and the debate was intense. Husserl was familiar with the problem and technically prepared to contribute.

3 The Multiple Tasks of a Philosophy of Space

For Husserl, there are three main groups of questions concerning a philosophy of space:

(a) Psychological questions: Is our representation of space intuitive (either adequate or non-adequate) or conceptual? If it is conceptual, is it founded or not on intuitions? There also belong to this group questions concerning the origin and development of spatial representation as an adaptive response to the environment (a natural history of spatial representation, so to speak). Husserl distinguishes two complementary tasks for these analyses: that of *descriptive* psychology, which is to describe the primary content, including the basic relations, of the spatial representation; and that of *genetic* psychology, which is to determine from which elements, by which psychophysical functions, according to which laws the spatial representation originates. For him, these tasks are not independent; descriptive analyses provide the foundation for genetic analyses.

(b) Logical questions: Does our representation of space serve cognitive purposes? If it is intuitive, is it adequate for understanding? If it is conceptual, is it logically founded (i.e. logically and epistemologically *justified*)? In short, is our representation of space, be it intuitive or conceptual, appropriate for knowledge? Is the intuition (or concept) of space an adequate foundation for the science of space? Are geometrical ideas and assertions a priori or a posteriori? Are geometrical concepts only idealizations of empirical concepts? Does geometry require intuitive procedures? Is the intuitive preferable to the symbolic in geometry? Are geometrical procedures intuitive by ostensive constructions or non-intuitive by pure conceptual analyses? Is geometry an inductive science (in a deductive format) or deductive by nature?

(c) Metaphysical questions: Does our representation of space have metaphysical value, i.e. does it correspond to something transcendently real or does it possess only "valid foundations" on something whose essence escapes us? Is pure geometry only a convenient fiction, despite its appropriateness, when conveniently interpreted, as a theory of physical space as we represent it? Is our spatial representation true to *form*, but not *content* (i.e. formally, but not materially faithful to reality)? For Husserl, logical and metaphysical investigations are intertwined; the former have fundamental relevance for the latter.

As a starting point, it is interesting to see what, for Husserl, are the main features of some of the most important philosophies of space. He reminds us that, for Kant, space is an a priori intuition, a condition of possibility of knowledge possessing empirical reality, but transcendental ideality, and whose metaphysical correlate is nothingness. The propositions of geometry are synthetic a priori; its method is the ostensive construction; and it applies to experience due to the a priori character of the spatial representation. Although Kant's is obviously a theory to be reckoned with, Husserl does not show signs of uncritical alignment with any of these views.

For the perspective Husserl calls positivism, he tells us, there is a difference between empirical and pure space, and only applied geometry has metaphysical relevance. Positivists recognize, *correctly*, according to him, the *necessity*, founded on the thing itself, of geometrical judgments. For them, Husserl tells us, empirical space is an idealization of intuitive space, and there are no reasons to suppose it has a correlate in reality. From what Husserl says here, one thing stands out—he thinks geometrical assertions are necessary (which, as we will see, does *not* mean that they are independent of experience, only that experience *cannot falsify* them). This is clearly in dissonance with views such as Helmholtz's or Poincaré's (if we take these views as referring to intuitive, pre-scientific space, which was not their original intention). Husserl thinks otherwise, that *intuitive* spatial representation and the truths based on it are necessary; not because, in line with Kant, they are independent of experience, but because our perception of space, with the properties pertaining to it, is the *only* response to spatial experience we managed to develop throughout our biological, psychological and cultural history. For Husserl, we are not free to *intuit* space in any other way; not even exotic spatial experiences can force us to modify our intuitive representation of space. If science decides, as it very well can, in favor of a spatial representation differing from the intuitive, it will do so by degrading the epistemological primacy of intuition.

For realism, he says, geometry is an instrument of natural sciences, of which space is a fundamental presupposition. For this perspective, however, Husserl tells us, space is neither representative (i.e. intuitive) nor idealized, but simply a tri-dimensional continuum. It seems that here, for science, Husserl has in mind mathematized or, as he calls it in *Crisis*, "Galilean" science of nature, in which space as experienced is "modeled" or, rather, substituted by abstract, purely formal "spaces". For realists, Husserl continues, geometry itself is a natural science, the *abstract* science of real space.

For Lotze, Husserl says, the facts that space is tri-dimensional and "flat", i.e. has constant zero curvature, are *logical* necessities (and the corresponding assertions, *analytic*). For Husserl, and this is an important aspect of his philosophy of space, these claims are *false*. The number of dimensions of space and its curvature, Husserl claims, are *empirical* facts (a posteriori). Only experience could teach us about these things.[2] I will have more to say about this below.

[2] But can we say from experience that space is tri-dimensional "all over"? As we will see below, Husserl seems to admit that, although we cannot access every corner of reality intuitively, we extend to the space

4 The Many Concepts of Space

There are, for Husserl, different conceptions of space, forming, in his words, a genetic progression: the space of our quotidian life, which we know perceptually; the space of pure geometry, accessible only to geometrical intuition; the space of applied geometry, that of natural science (before, of course, science got bold enough to consider other representations of physical space, which happened a few years after Husserl wrote his notes); and the metaphysical or transcendent space. The space of quotidian life is, for Husserl, neither simply an imprint of external space on consciousness, intermediated by the relevant sensorial systems (visual, tactile, kinesthetic)—we do not simply abstract space from data of sensorial perception—, nor, contra Kant, an a priori mold we impose on outer experience. Intuitions are the raw material from which we actively *constitute* a representation of space, without the help of any constituted body of knowledge, mathematics or physics, particularly—the constitution of a perceptual representation of space is not based on scientific knowledge, *it takes place at the pre-predicative level of consciousness* (which, it is important to notice, is not, for Husserl, a domain of pure passivity).[3]

The space of pure geometry is an altogether different matter. It is a mathematical domain ruled by concepts that, regardless of their origin, only non-sensorial geometrical intuition, which despite its roots in perception is non-perceptual, is equipped to investigate. The space of applied geometry is the physical space as we represent it taken as a model (or an approximate model) of pure geometry, and, finally, transcendent space is space as it really is, not simply as we represent it to be.

The intuitive space is the first in the genetic progression; then comes the space of applied geometry, which is intuitive space taken as an approximate model of pure

Footnote 2 continued
as a whole the structural properties perceived in our limited chunk of space. Space *as represented* is, for him, a *uniform* extension of space *as directly intuited*. There are no reasons, presumably, for our space-constituting functions to work differently if we were placed in another region of space. The fact that physicists conjecture today that space may have more than three dimensions in subatomic scale does not pose any treat to our *pre-scientific* representation of space; science, Husserl believed, has the right to adopt any *formally correct* explanation of observed phenomena (even incurring in the risk of formalist alienation, as denounced in *Crisis*).

3 There is in Husserl a clear distinction between pure sensorial data (the hyletic data) and percepts (perception is an *intentional* experience). In his Lectures of 1907, Husserl presents a minute description of the constitution of spatial rigid bodies, and the medium where they are placed, physical space, in which we are not allowed to forget this distinction. Husserl believes there are essentially two systems of sensorial data involved in spatial perception, the visual and the tactile (although, in that work, the visual appears with by far more relevance), which are molded into spatial percepts by a series of *intentionally motivated* kinesthetic systems working in isolation and cooperatively; there are basically four of these systems (some terms are Husserl's, some are mine): (1) the oculomotor system, by means of which a non-homogeneous 2-dimensional flat finite space is constituted; (2) the restrict cephalomotor system, by which a non-homogeneous 2-dimensional *curved* space is constituted, limited "above" and "below" by closed lines, like the section of the earth surface between the tropics; (3) the full cephalomotor system, by which a 2-dimensional spherical space is constituted (which Husserl calls Riemannian space); and, finally, (4) the (full) somatomotor system, by which Euclidean space is constituted. It is worth noticing that, for Husserl, binocularity does originate depth, but depth is not yet, by and in itself, a third dimension comparable to breadth and height; spatiality requires the subject to be able to move *freely* towards, away from and around the body, and would be constituted independently of binocularity.

geometry (the space of applied geometry is then a theoretical enlargement of intuitive space); next the space of pure geometry, obtained by idealization from intuitive space; and finally, metaphysical space, which is an object of pure thought. Each step in the genetic sequence of spaces takes us further from sense perception. Parallel to the sequence of spaces runs that of constituting acts, beginning with raw perception, followed by theoretically enriched perception, then idealization, ending with non-intuitive (empty) conceptualization. There is also the possibility of generating other conceptions of space by free-varying geometric space in imagination. In this manner we can obtain all sorts of spaces of mathematical interest, n-dimensional Riemannian spaces in particular (the word “Riemannian” here refers to abstract formal spatial structures, not elliptic or spherical spaces, which are, like the Euclidean space, particularizations of the generic Riemannian space).

In Husserl’s own words, the following are the questions he wants to answer (it does not hurt to remember): based on what content can the intuition of space offer a foundation for pure geometry; in particular, how can geometrical intuition have a basis on *perceptual* spatial intuition? To what extent can we give an *objective* meaning to geometry and a transcendent epistemic value to geometrical assertions (i.e. are geometrical truths true also of transcendent, not only geometric space)? Does (geometrical) intuition have any metaphysical correlate (i.e. does *geometrical* intuition correspond to anything metaphysically, that is, transcendently real)?

Let us take a closer look at these different conceptions of space:

(a) For Husserl, *intuitive* or *represented* physical space (physical space as we represent it) is more than what meets the eye (or any sensorial system for that matter). It can be better described as the resultant of sensorial impressions (visual, tactile, kinesthetic) and *innate* and *fixed* psychophysical functions whose task is to accommodate these impressions into a coherent spatial mold. Our senses offer the raw material that built-in systems (selected in the course of our biological history) put in spatial form. Our retinas, for example, offer two slightly different two-dimensional projections of the multiplicity of outer things (the difference being the binocular parallax); our perceptual system (which for Husserl, remember, involves *more* than the purely sensorial) eliminates double images by “creating” a perceptual “depth”. But we do not have to go into these matters here; it is enough to keep in mind that, for Husserl, our intuitive spatial representation is *neither* completely prior to sensorial experience, *nor* abstracted completely from it; the subject has an active role in constituting space from raw spatial perception.

Space is represented first and foremost as a multiplicity, an extended medium where data of *actual* or *possible* spatial perception—bodies and bodies *in spatial relations*—must be accommodated. We intuit space by intuiting things *in space*; space and the spatial are never represented in themselves, but as *dependent moments* of intuited things (we cannot, for instance, see space, but things in space; and if we see, for example, body B *between* bodies A and C, we *only* see this as an *aspect* of the spatial complex formed by A, B and C). Moreover, any spatial extension is a *part* of space, the encompassing totality. Although Husserl does not say it explicitly, it follows from this that we represent space as *one*, not many, and, moreover, in one piece (a *connected* manifold, i.e. all positions in space are spatially related to each other). This is an a priori character of our representation of space, not because it belongs analytically to the notion of a spatial manifold, but because the manifold of actual and possible spatial *intuitions* (the spatial hyletic data), which the spatial manifold must *in-form*, is a manifold connected around a *center*, the perceiving I. Any data of the outer senses (even if they belong to different sensorial systems) must stand in some spatial relation to each other simply because they are all perceptions of *the subject*.[4] Any spatial extension, Husserl says, is in space as a part, a limit (points, lines, surfaces) or a connection or relation among parts (for instance, the space *between* spatial extensions).

Space, Husserl also says, is conceived as *infinite*, that is, ultimate parts of space are not *thinkable*. In fact, *finite*, but *closed* manifolds (like the surface of a sphere) do not have ultimate parts either; so, to be rigorous, Husserl should have said that we conceive space only as *unbounded*. However, as we will see below, since for him space is also represented as torsionless (zero curvature at any point), it must necessarily be infinite. It is worth noticing that space cannot be *intuited* as infinite; we can only intuit *limited* parts of space, but also, simultaneously, *conceive* an indefinite enlargement of intuited parts, since beyond actual intuitions there is an open *horizon* of possible intuitions.

For Husserl, on the other hand, only experience could have shown us that physical space is *continuous* and *tri-dimensional*[5] (even though, as he admits, discontinuities in transcendent space could exist, undetected by the senses). Nonetheless, he thinks, our space-constituting functions are not biased towards continuity or tri-dimensionality; we, remaining as we are, could have come up with a different representation, a space with a different number of dimensions or discontinuities, if experience had led us that way.[6]

What about *flatness* or absence of torsion? We *could* have concluded that space is flat by verifying that the sum of the angles of any arbitrary triangle was equal to two right angles (Tales' angular theorem, which is equivalent to Euclid's fifth postulate).[7] The problem, of course, is that physical space is not geometrical space;

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[4] The contemporary scientific image of space is that of a system of formal relations induced primarily by *physical* relations among physical entities (independently of how *we* happen to perceive or conceive it). It follows that if these entities were separated in clusters, with entities in one cluster having no physical relation to those in other clusters, space could very well be conceived, for scientific purposes, as disconnected into isolated "multiverses", bearing no *spatial* relations with one another (logical relations such as that of difference, would, of course, still hold).

[5] We could in principle conceive different organisms, with different space-constituting mechanisms, representing space with a different number of dimensions (for example, bodies incapable of motion might represent physical space with only two dimensions—see Husserl 1997 or note 3).

[6] It is also conceivable, I think, that the psychophysical functions responsible for the representation of physical space do in fact make it impossible for us to perceive spatial discontinuities, even if they existed in transcendent reality. The situation might be analogous to us representing movement in fast running discrete sequences of still pictures: closely packed atoms of space might necessarily be perceived as a continuum. But this seems not to be what Husserl thinks (although I'm not willing to bet on it).

[7] There are other methods, such as trying to draw a square by drawing congruent segments at right angles; only in Euclidian space this will succeed.

there are no straight lines, perfect triangles or precise measurements in it. However, we might insist, if our measurements, imprecise as they are, oscillated around the "precise" value of two right angles, so that in the average Tale's theorem were approximately true, this might suffice to convince us of the Euclidean character of space. But, for Husserl, our intuitive representation of physical space is pre-scientific; mathematical or scientific considerations have nothing to do with it. How can it be then that we *do* represent space as a *flat* manifold?

It seems safe to suppose that we know from experience that rigid bodies exist (remember that, for Husserl, we do not have intuitive access to space directly—spatial properties are abstract properties—, but only via bodies in space).[8] A rigid body is one that changes neither shape nor size (which are morphological notions given perceptually) by simply moving in space (without some *physical* reasons for so doing, excluding, of course, the action of space, which by itself is causally inert).

Well, one can object, we *seem* to know plenty of bodies that are for all practical purposes rigid. But how can we be *sure* of this? They may alter shape or size as a consequence of motion without us being able to notice it. If bodies apparently rigid, including our bodies and standard meters, deformed in motion uniformly, we would not notice any deformation by strictly geometrical means.[9] We, however, do not have to be *sure* that apparently rigid bodies are *really* rigid; our representation of physical space is not based on sound knowledge, being as it is a pre-reflexive response to experience. We do not even raise the question; if bodies *appear* rigid, rigid they are. We simply *experience* rigidity, even if this experience did not correspond to anything real.[10] Moreover, since rigid bodies can move freely without changing shape or size (because, precisely, they are rigid) they can be compared with respect to shape and size: it is enough to superpose or just bring them close to one another and verify. The notions of congruence—same size and shape—and similarity—same shape, not necessarily same size—are then purely intuitive morphologic concepts; bodies in physical space can be *perceptually* related to each other in terms of (approximately) the same or different sizes or shapes.

Now, we know that the existence of rigid bodies does not necessarily imply that space is flat (without torsion or curvature at any point); it only tells us that it has *constant* curvature, zero, positive or negative (Helmholtz-Lie theorem). But if space had a positive or negative curvature *noticeable within the range of immediate perceptual experience*, our experience of (bodies in) space would have been different from what it is (the difference being a function of the degree of curvature; in an extreme case we would be able to scratch our backs by stretching our arms in

[8] "There is no doubt that the conviction which Euclidean geometry carries for us is essentially due to our familiarity with the handling of that sort of bodies which we call rigid and of which it can be said that they remain the same under varying conditions" (Weyl 1963, p. 78).

[9] Helmholtz, before Einstein, told us that the metric is not a formal aspect of space, but depends on its material content. So, even if space were not homogeneous and its metric not constant, free mobility would still be valid, since, in Weyl's words, "a body in motion will 'take along' the metric field that is generated or deformed by it" (Weyl 1963, p. 87).

[10] But what sense could be attributed to the hypothesis that the world and everything in it change dimensions in such a way that no change can in principle be noticed? As Weyl tells us (1963, p. 118), a metaphysically real difference that cannot *as a matter of principle* be detected is non-existent.

front of us!). We then know from experience that space, when its structure is spelled out in geometrical terms (something we do *not* do in ordinary perception; the perceiving subject, simply *as such*, is not necessarily a reflective subject) is either *locally* flat or only slightly non-flat (so slightly we wouldn't notice it). Either possibility suffices to justify *empirically* our pre-scientific representation of physical space *as a whole* to be flat, which is all we need (even, as Husserl admits implicitly, nothing can guarantee we would not observe torsions or deviations from flatness if our senses were more refined). In short, for Husserl, our *perception* of space requires us to represent it as flat.

Helmholtz, among others, admitted that by putting the structure of space to empirical testing it *could* be perceived as non-Euclidean. Husserl is at odds with this view. Although he believed metaphysical space could, as a matter of *fact*, be non-Euclidean, and science could represent it so (even if it *weren't* in fact so), he didn't think we could ever *perceive* space as non-Euclidean. Helmholtz fails to see, Husserl claims, the distinction between *physical* and *psychological* experiences of space. In the latter the mind is at work making *sense* of (or bestowing sense on) sensorial experiences; for us, the *perception* of physical space only makes sense in the Euclidean mold (Husserl is of course pointing here to the difference between *passive* and *active* experiences, an important distinction in genetic phenomenology that he will make explicit in later works).

Summing up; for Husserl, physical space can be *represented* as a continuous connected tri-dimensional infinite flat manifold (as well as homogeneous and isotropic). Our representation of space, however, may not correspond in some respect or another to transcendently real space. Husserl makes a strong claim in this respect; for him, congruence and *all* geometrical properties seem to be in the nature of our *sensations*, not in transcendent reality. Our geometrical-constituting functions, like our tone and color-perception systems, may then give us something that is not strictly speaking out there. Transcendent space may be only an analogue[11] of represented space. But, be as it may, Husserl believes *we* are *empirically justified* in representing space the way we do; our representation of space is *objectively*, even if not maybe transcendently real.

It is an important question which properties of space belong *necessarily* to it, irrespective of experience or the psychophysical functions involved in the representation of space; that is, what belongs analytically to our conception of space. From what Husserl says, he believes that any space must necessarily be conceived as a manifold, which only means that it has parts or components.[12] Once a manifold is given whose parts entertain relevant relations with one another, as is the case of space, it is natural to think that any part of the manifold can be univocally determined in terms of some relations to some of the other parts (for example, any sound can be determined by its pitch, intensity and timbre, which are relational determinations) even if this determination were not uniform throughout

[11] It is tempting to read this analogy in terms of the notion of homeomorphism. Represented space may be only a homeomorphic copy of transcendent space; i.e. only partially and, even so, only formally identical to it.

[12] A manifold is no more, no less than a structured multiplicity of things; we would call it today a structured system of entities.

the manifold; i.e. some notion of dimension must be available a priori, even if we think it as varying from region to region. Husserl is not clear on this point; he only insists that the three dimensions of space are empirically given. But I think he would agree that a manifold can be thought as finitely or infinitely dimensional, with constant or variable dimension, but not as dimensionless.

As already observed, Husserl claims that space cannot be finite, for we cannot *think* of a space with ultimate parts. So, for him, infinity of space (although, as I have pointed out, this argument can only conclude that space is unbounded) is also given a priori; but in a different, *transcendental* sense of a priori. What Husserl has in mind (besides Kant's antinomies) seems to be this: since space must accommodate the totality of all possible experiences (we only have one space for we have only one transcendental subject) and we can always conceive extending the limits of our field of experiences, space cannot be bounded. Infinity is a priori not because any spatial manifold must be so conceived, but because our (or the transcendental subject's) field of possible experiences must. For reasons already explained, an analogous sense of a priori applies to connectivity. The absence of torsion (constant curvature) and the three dimensions of space, on the other hand, are for Husserl a posteriori, in more senses than one: *neither* the notion of space, *nor* our space-constituting functions per se, independently of *complementing* sensorial data, require them. If raw *physical* experience so induced us, Husserl suggests, we would have represented space as four-dimensional or curved. A fortiori, it is not true that any sentient creature in this world that can develop a representation of space will represent it as flat or tri-dimensional (more dramatically, we shouldn't expect extraterrestrial beings to share our geometry: fluid beings in a fluid world would probably have no concept of rigidity, similarity or congruence, and no Euclidean geometry either). Space is represented as tri-dimensional and torsion-free due solely to experience. On the other hand, indefinite divisibility of space seems to follow necessarily from the infinite extensibility of experience, being in this sense also transcendentally a priori.[13]

Now, can experience induce us to change our representation of physical space?.[14] At first sight Husserl seems to be forced to accept that it can; after all, experience can take what experience gives. But, from what has just been said, although the psychophysical functions responsible for the constitution of our spatial representation require, in order to perform their task, an input of sensorial data, they cannot, for Husserl, be restructured by incoming data, being as they are products of our adaptation to the environment. Given that they developed to function in a certain way, they will function in that way no matter our physical experiences (what does not mean that in the course of eons radically new experiences couldn't give our descendents altogether different space-constituting functions, long after we, their no-longer-spatially-adapted ancestors, vanished from earth). Effects of perspective

[13] This is the transcendental synthetic a priori in Husserlian version.

[14] In answering this question Husserl seems to be answering Helmholtz, who argued *against* the non-intuitability of a non-Euclidian space. For Helmholtz, to intuit a non-Euclidian physical space means to *imagine* spatial sense impressions, captured by our sense organs according to the known laws, which would force a non-Euclidian character on space. As I show in the main text, Husserl explicitly denied this possibility.

and like phenomena, for instance, Husserl says, are not seen as distortions of space, but visual *illusions*. If *we* remain what we are our representation of space will not change, or so Husserl thinks.

Again, this can count as an argument against Poincaré's style of conventionalism (if taken to apply to intuitive space) according to which we are to a large extent free to choose this or that spatial mold for our intuitions, for experience by itself is not compelling. Husserl on the contrary believes that our representation of physical space, the one we *effectively* possess, is the only we can *perceive*, even if it is not the only we can *conceive* (in fact, as *thinking*, and not only *perceiving* subjects, we may even decide *against* perception in our scientific representation of space). We may be wrong as to how physical space *really* is, but if we were, we are not free to be right.

What about *radically new* experiences, light rays bending in some region of space without any physical reason for so doing or, more prosaically, otherwise rigid bodies behaving in the strangest ways; could they force or suggest a change in our spatial representation of the outer world? Again, Husserl answers in the negative. In fact he explicitly says that if the field of vision were altered, we would say this was no longer the field of vision, but a new experience; no longer space, but something new (drug induced "trips", with lots of spatial distortions, such as the melting of solids, as a colleague in Berkeley—where else?—once reported, may be interpreted by a marveled drug (ab)user as an enlargement of the "doors of perception", giving him access to another space, but more likely he will take this for what it is, an hallucination, a *mis*perception of reality); our concept of space is not altered, but new experiences, new concepts would be generated (suggesting maybe a different *scientific* representation of physical space). We can also, of course, *always* hypothesize some physical reason to account for the strangeness of observed phenomena; we can even consider this behavior as *evidence* for some unknown physical state or process.

But as Einstein has shown, this is not always the best (or methodologically sounder) solution. Things may be simpler if we just change our conception of space. This is what Poincaré had in mind in saying that applying geometry involved some degree of arbitrariness. History of science has apparently shown Husserl to be wrong on this particular aspect: we *have changed* our representation of physical space, pressed not only by experience, but by *reason* too. But, I must insist, the representation of physical space Husserl is concerned with, as said before, is *not* the scientific representation of physical space; representing space intuitively is prior to any scientific reasoning. Husserl would not disagree that science could for higher reasons exert some violence on our pre-scientific representations. One of the *logical* and *metaphysical* problems posed by a philosophical investigation of space, as mentioned above, is precisely whether our pre-scientific representation of physical space does serve cognitive purposes or has a transcendent correlate. We may realize that it does not and change it accordingly; but we would not ipso facto change our way of *perceiving* space.

We ordinarily say, and Husserl also said, that intuitive space is the realm *par excellence* of the *a peu près*, the approximate, the morphological, the organic (representing physical space perceptually involves *abstraction*, but not *idealization*). But, as Husserl himself noticed in *Crisis*, this is so only by comparison with its ideal counterpart, geometrical space. Intuitions are what they are, and they are not stable vis-à-vis geometrical ideals: they oscillate around ideal points and can only be said to be approximate if we take these points as the "truly real" reality, something Husserl criticized in his later period as a sort of alienation.

Represented physical space is not geometrical space; it does not contain geometrical points, perfectly flat surfaces or perfectly spherical bodies. But it is not devoid of structure. Physical space, as we represent it, is a proto-geometrical manifold with morphological counterparts to most geometrical structures and relations (proto-metrical and proto-topological relations, in particular). Spatial extensions, Husserl says, can be compared as to shape and size. Relations such as equal (or similar) and unequal (or dissimilar) with respect to shape, and larger, smaller and equal, with respect to size, can be established among spatial extensions. But these are *not* relations of *measurement*. Relations of equality or inequality, of shape or size, are based simply on direct *perception* (not measurement). There can be inequality without gradation, Husserl says. We can compare spatial extensions as to size and shape without measurements, to the same extent we can perceptually establish a topological relation of order among them (region A is "approximately" *between* regions B and C).

(b) *Geometrical* space is the idealization of intuitively perceived physical space (perceptual space as we represent it to be) taken as a mathematical manifold in its own right. Pure geometry, the science of geometrical space, becomes physical geometry when we take represented space as a (n) (approximate) model of it. In the strictest sense physical geometry is a mathematical theory of nature, such as dynamics or electromagnetism. Weyl, who often endorsed Husserlian ideas on the nature of mathematics, said (Weyl 1918) that a mathematical theory of some intuitive concept (such as that of continuum) often fails to be completely faithful to intuition, and is at best an approximation whose epistemological value must be put to empirical testing. Husserl seems to agree. Science can force us to review our representation of physical space; physical geometry (but not pure mathematical geometry) may be proved wrong, like any theory of nature, mathematized or not.

For Husserl, geometrical space is a structure logically elaborated on the basis of the spatial intuition of pre-scientific consciousness, in which nothing can any longer be said to be perceptually represented or capable of being represented, but only that it is thinkable. The geometrical intuition geometers refer to is, according to Husserl, only an ideal; geometrical truths cannot, properly speaking, be perceived. *Perceptual structures* (diagrams, drawings, etc.) can only stand as *symbolic representatives* to geometric structures, for geometric intuitiveness *cannot be perceptually realized*. Perceptual representatives, for Husserl, represent by perceptually displaying truths that stand in a relation of *analogy* to truths of pure geometry. In other words, even though we *cannot* intuit geometrical truths in the diagrams we draw, we can perceive therein truths standing to geometrical truths as symbols to the symbolized. When a diagram is drawn and taken to represent (as a symbol for) an abstract structure that is not seen (and cannot be seen) something is perceptually presented that serves as a symbol for something not perceived (but not conceptualized either). Geometric intuitiveness rests on this relation of analogy.

This last remark is worthy of some comment. If geometrical notions were purely conceptual, geometry would be a conceptual science where only conceptual intuition would have a place; ostensive constructions in intuitive space would have none or little relevance. But, as Husserl believed, geometrical structures and concepts originate by idealization from mereological analogues (some philosophers, Husserl says, do not accept this because they do not see how idealization can be possible without an ideal). Therefore, ostensive constructions, although not geometrical in the proper sense, have some bearing on geometrical truth. There is a relation of analogy between perceptually displayed truths (concerning represented space) and non-perceptual (and non-conceptual) geometrical truths: what is *shown* in the former (concerning mereological structures) is *thought* in the latter (concerning geometrical structures). This is why we can, in particular, claim an intuitive foundation for geometrical axioms.

On the basis of actual perceptual intuitions (in actual perception or imagination) we can, for Husserl, posit by *idealization* what is not perceived, that is, the unreachable (and not amenable to perception) limit of a sequence of perceptions: geometric points as limits of sequences of vanishing spatial regions; geometric lines as limits of sequences of narrower and narrower perceptual lines; geometric surfaces as limits of sequences of thinner and thinner perceptual surfaces. As explained above, for Husserl, we can reason about objects that are only “emptily” represented by reasoning about their perceptual representatives, provided we do not allow “irrelevant” properties of the representatives to interfere, that is, properties that do not play any representational role. We must, Husserl says, distinguish between (1) the actual intuitions given in perception or imagination and (2) the intuitive procedures depending on intuitive signs that justify the (judicative) content of certain judgments that are *not* intuitive. Here, Husserl claims, intuition and thinking are intimately connected.

Not only the fundamental geometric elements—point, line and plane—are idealized from perceptual correlates; the morphologic notions of (approximately the) same size and (approximately the) same shape also generate, by idealization, the geometric notions of similarity and congruence: two geometric structures are congruent (resp. similar) when they have *exactly* the same size and shape (resp. same shape). The notion of congruence captures that of constancy of size and shape independently of place (rigidity); that of similarity the notion of invariance under change of scale (similarity, not congruence, is the quintessentially Euclidean notion). The morphologic notion of order also gives origin, still by idealization, to the correspondent ideal mathematical (topological) notion of order: point A lies *exactly* in between points B and C. Having points, lines and planes as basic elements, and the relations of belonging, congruence and order (betweenness) as the fundamental notions, Husserl sketches an axiomatization of geometry that is in the main lines identical to Hilbert’s[15]:

[15] All the definitions (and subsequent assertions) are to a large extent Husserl’s own; I only tried to give the ensemble a more coherent (although not logically flawless) presentation, remaining as close as

Definitions:

(D_1) any two points A and B determine a *segment* (denoted by AB); a point C *lies* on AB if, and only if, C is between A and B, C = A or C = B (this definition coincides with Hilbert's). If A = B, AB is a null segment. There is only one segment determined by A and B, which can also be called the *distance* between A and B (this is not yet the *numerical* distance). The distances between A and B and between C and D are *identical* when the segments AB and CD are congruent.
(D_2) any two *distinct* points A and B determine a *straight line* (line through A and B). C lies on line through A and B if, and only if, one (and only one) of the following holds: C is between A and B (C lies on the segment AB), B is between A and C or A is between B and C. If either B is between A and C or A is between B and C we say that C lies on a *continuation* of the segment AB. Obviously, there is only one line through A and B (*the* line determined by A and B). The definition Husserl actually gives is the following: any two points A and B determine a straight line (line through A and B), such that a point C lies on this line if, and only if, AC has either the same or opposite direction with respect to AB. This definition, however, depends on that of direction.
(D_3) two straight lines are *parallel* when no point lies on both lines.
(D_4) any segment AB can have two possible *directions*, AB or BA, depending on the ordering of A and B. A *direction* is then, for Husserl, a segment in which an ordering was established, i.e. an ordered pair of points. It is not difficult to define equality of directions in terms of our basic notions. If segments AB and CD are not collinear, directions AB and CD are identical when the straight lines through A and B and through C and D (call them *a* and *b* respectively) are parallel and the straight line through B and D does not meet the straight line through A and C in a point in the plane region delimited by the lines *a* and *b* (which is the set of points of the plane lying on a segment with endpoints in *a* and *b* respectively).[16] If AB and CD are collinear, they have the same direction if there is a segment XY not collinear with them that is co-directional with both (see A_4).

The following assertions are, for Husserl, *intuitively* justified:
Assertions:

(A_1) there is a unique segment between any two points A and B (see D_1). This is Euclid's first postulate.

Footnote 15 continued
possible to Husserl's original ideas. Although Husserl's approach to the axiomatics of geometry is remarkably similar to his colleague Hilbert's, in his famous axiomatization of 1899, it was for the most part developed before they became colleagues in Göttingen, in 1901. But this can be explained: both were buds of the same Paschian branch. Recall that also in Hilbert's system the basic elements are point, line and plane, and the fundamental relations those of incidence (point lies on line or plane, line lies on plane), order (betweenness) and congruence. The notion of continuity (which in Hilbert's system is given by the axiom of completeness and the Archimedean axiom) does not appear in Husserl's sketchy system explicitly.

[16] I didn't find this definition in Husserl, but I think he wouldn't object to it. It may also be that he thought sameness of direction was a primitive notion.

(A_2) any segment can be indefinitely and uniquely continued (see D_2). This is Euclid's second postulate.
(A_3) from any point A there are identical distances (equal to the distance AB) in any direction (i.e. there is a circle with radius equal to AB centered at A). This is Euclid's third postulate.
(A_4) given any direction AB and any point C, there is a point D such AB and CD have the same direction (changing "direction AB" into "segment AB" and "have the same direction" into "are congruent", this assertion bears close resemblance to Hilbert's first axiom of congruence). Since sameness of directions, in the case C does not lie on the line through A and B, implies that there is a line through C parallel to the line through A and B, this asserts in particular that given a point C and a line not through C, there is a parallel to this line passing through this point.
(A_5) distances and directions are mutually independent (points at equal distances can determine segments with different directions; segments of same direction can be determined by points at different distances).
(A_6) any distance (or segment) can be divided in equal parts.

It is not difficult to see that in developing Husserl's ideas in a logically satisfactory way (for example, by stating explicitly the axioms of incidence, order, and congruence, among others things) we could come up with a satisfactory axiomatization of Euclidean geometry along Hilbertian lines—one difference between both is that Husserl seems to privilege the notion of direction. Husserl himself does not go that far, but from what he says it is already clear that, for him, it is possible to develop pure Euclidean geometry on intuitively justified axiomatic bases (where, of course, intuition here means *geometrical* intuition).

(c) *Mathematical spaces*: these include geometrical space considered from a purely mathematical perspective and all its free variations, the *formal* structures mathematicians invent often pressed by necessity or the internal development of mathematics. For Husserl (*Logical Investigations*), *pure* mathematics, including *pure geometry*, is part of formal ontology, the investigation of formal structures in which arbitrary objects, no matter which, could in principle be accommodated. Mathematical manifolds, i.e. collections of "points" (abstract forms of unspecified objects), can be finite or infinite, discrete or continuous, with or without a topology or a metric.[17] Any manifold has a dimension, uniform or variable, for as I argued before dimension seems a necessary aspect of any mathematical space. Riemannian spaces are *continuous* n-dimensional spaces in which points have coordinates and a local metric is defined by a quadratic differential form so as to render the lengths of any two line segments commensurable to each other. There are also non-Riemannian spaces (Riemannian spaces are not the most general spatial structures), in which the metric is even more general, and spaces with no metric al all.

[17] Discrete (finite or infinite) manifolds (nets) have a natural notion of distance: we can define the distance between two points as the smallest number of points one has to go through to reach one from the other. Non-discrete manifolds, continuous ones in particular, on the other hand, have no "natural" notion of distance and can accommodate various.

5 Pure and Applied Geometries

Represented space is our Lebensraum, the space in which we live, pressed by the necessities of life and survival. It is originally the Ur-space which radiates from the I (*my* space) and only derivatively a representation of *objective* physical space (*our* space). We experience objective physical space by experiencing bodies "immersed" in it. This experience eventually coalesces into a body of knowledge, a proto-geometry sufficient for the practical demands of life, the embryo of pure geometry. Pure geometry is the scientific investigation of the properties of space, no longer the space of perception, but an idealized version of it. In the process of idealization mereological concepts, real constructions and perceptual intuitions give origin, and are taken as symbols to, respectively, geometrical concepts, ideal constructions and geometrical intuitions. But as any mathematical discipline, pure geometry cannot be confined to the (geometric) intuitive realm. Projective techniques, for example, require points and lines at the infinite that are not amenable to intuitive representation. Even in the case of traditional geometrical methods, concepts may be introduced and geometrical structures considered whose manipulation cannot be accompanied by intuitions. The situation is analogous to arithmetic, where numerical concepts (such as those corresponding to large or "imaginary" numbers) and conceptual operations (summing these numbers, for example) are beyond the reach of proper numerical intuition, being only indirectly accessible via symbolic methods.

A question then imposes itself, a typical question at this moment in Husserl's philosophical development (middle of last decade of the nineteenth century): how can we be sure that *pure* geometry, when taken to represent space and *refer* to it, i.e. when transmuted into *physical* geometry (the mathematical theory of represented physical space) is (at least approximately) *true*? For Husserl, this is the most important *logical* question concerning pure geometry. This problem appears as naturally in geometry as in arithmetic, where it had already surfaced and been dealt with in a way Husserl found satisfactory (*Philosophy of Arithmetic* of 1891 and minor works dating from 1890 to 1900–1901, years of the publication of the *Logical Investigations* and the lectures given at Gottingen). See Husserl (1970b) and (1994).

Husserl says explicitly that we must show that all assertions concerning measure and position valid in pure geometry are also valid in physical geometry; valid that is approximately, in the sense that real numbers of pure geometry will become the rational numbers of physical measurements. How can we be sure, he asks, that the laws of the ideal science of pure geometry, including those more remotely or not at all related to intuition, are *correct* if taken to refer to physical space (as we represent it)? The axiomatic method is the answer.

Axiomatization was then, and by some time already, making a come back after having lost ground to analytic and projective techniques in geometry. This tendency reached its peak with Hilbert, the champion of the axiomatic method (who, incidentally, also explicitly claimed intuitive foundations for his axiomatic system of geometry, "a logical analysis of our perception of space", according to him). Husserl himself, as we have seen, sketched one such axiomatization, based on the intuitively grounded notions of incidence, congruence and betwenness. For him, the axioms of the system were, as we have also seen, *true*; *geometric* intuition tells us so. Since pure geometry is a *logical* system based on intuitively true axioms, which are approximately true for represented physical space, the theorems of pure geometry must also be true for geometric space and approximately true for represented physical space. Physical geometry is then epistemologically justified (Husserl also considered axiomatization as possibly the one way of putting general arithmetical systems on sound logical bases).

Axiomatic reasoning may or may not involve intuitions. Euclid's system relied heavily on perceptually grounded geometrical intuitions; proofs were often accompanied by diagrams (although they sometimes are only illustrative) or consisted basically in pointing to supposedly indisputable evidences. Hilbert's or Husserl's, on the other hand, confined intuitions to the axioms; the theorems of the system can only be guaranteed to be true (of *geometric* space) if derivations in the system are strictly logical: logical consequences of true axioms are necessarily true. Since represented space approximates geometrical space, pure geometry, axiomatized on logically sound bases, is sure to lead to approximately true facts about physical space as we represent it.

As we see, Husserl is definitely not an adversary of symbolic methods in mathematics, not even in geometry; he is not the "intuitionist" thinker he is sometimes depicted to be. He believes symbols have many advantages over intuition, including *heuristic* advantages. Equations, for example, he says, can make abstract relations intuitive. An equation is a relation among symbols that displays abstract relations *ad oculus* (in the sense that what is *shown* in the equation is *thought* abstractly, and may even be hidden in the abstract). As we have seen, for Husserl, drawings, diagrams and constructions in represented *physical* space are privileged geometrical symbols. But, to the extent that not all relations among algebraic symbols are relevant for the abstract content of equations, not every relation among empirical representatives are geometrically relevant. Although we can usually tell which relations are and which are not relevant, we can make mistakes. Axiomatization is a safer procedure; it confines geometric intuition to the axiomatic bases, and from that point on the responsibility for the production of knowledge falls on the shoulders of logic; we cannot go wrong. Although diagrams and other visible symbols can be useful, the symbols of a well-designed symbolic axiomatic system are, for Husserl, better.

6 Conclusion

If my reading is correct, Husserl's philosophy of geometry forms a coherent whole that can be summarized thus: there is an inner and an outer experience; the objects of outer experience are localized in space, which is not an a priori form of intuition; space exists transcendently. Spatial structure cannot be directly accessed; it must be intermediated by our experience with spatial objects. But it does not offer itself readymade to consciousness, through the senses, as an imprint; we must *constitute* a representation of space. Through our biological, mental and cultural history we managed to develop elaborate physical and psychological functions for this purpose.

Space-constituting functions cannot be modified at will, and the system of spatial relations they provide is in this aspect analogous to color or tonal systems. Experiences that do not fit into this system are not interpreted as spatial experiences.

We must necessarily represent space as a manifold, arguably endowed with some notion of dimension, and there is not much else logically required by the notion of space. Bodies in space stand in relations we call spatial; they can be far or close, get farther or closer (they can move), be bigger, smaller or equal to other bodies, be located between other bodies, etc. Space is the condition of possibility of spatial relations; it must be such as to "accommodate" all conceivable bodies and all the spatial relations they can possibly entertain with one another. Everything that bears the mark of exteriority happens in space. So, space must be such as to make this possible. These are what I call the transcendental a priori properties of space (it is clear to me that Husserl recognizes them as such). Since the complete field of all possible experiences of the outer senses is infinitely extendable and capable of indefinite refinement space must be unbounded (in the limit infinite) and infinitely divisible (in the limit continuous). Space is also unique; there is only one space, for there is only one field of all possible experiences of the outer senses.

We are not free to represent space as we like or find convenient, to the same extent we are not free to have a different color or tonal system. Husserl is not to any extent a conventionalist, at least as far as *represented* space is concerned. Contra Helmholtz or Riemann (if somehow unfaithfully interpreted as referring to represented space) he thinks the representation of space does not harbor hidden presuppositions or hypotheses. There are no hypotheses where our experiences and space-representing functions suffice. Contra Kant he doesn't think intuitive space is an a priori form of sensibility, and contra the empiricists he doesn't believe space is extracted in full gear from raw perceptual experience.

The representation of external, physical space we develop on the basis of psychologically (he'll say later intentionally) elaborated experience is of an infinite connected continuous flat tri-dimensional manifold, in which certain relations are immediately perceivable, in particular congruence, similarity and order. Represented space is the material from which higher "intentional" acts, idealization particularly, extract the space of pure and physical geometries (pure geometry is a mathematical science, physical geometry is pure geometry interpreted as the geometry of physical space). By idealization, perceptual intuitions can be refined into geometric intuition, on the basis of which the axioms of pure geometry are justified. Since geometrical truth flows down the line of axiomatic reasoning, we can be sure that applying pure geometry to physical space leads to truth, at least approximately.

Science may find reasons to choose different representations of physical space; but this, of course, does not mean that we will ipso facto *perceive* space differently. Only if we, particularly our psychophysical functions, and the way reality filters through our senses had been different we might have represented space in another way. By choosing space representations that may seem artificial from the point of view of our intuitive spatial representation, however, science may dig a gap between perception and reason that, *if not properly understood and adequately interpreted*, may alienate men from his cultural milieu. The mathematical "refinement" of the intuitively given can induce us to give ontological priority to what is phenomenologically secondary; only phenomenological analyses can establish the correct order of foundation, relocating the subject and its intuitions in the fundamental place they should occupy by right. It is not perception that is irremediably short of “capturing” reality as it is, but the idealization of reality that, by substituting reality as experienced perceptually with a mathematically purified version of it, puts the now idealized reality *in principle* out of the reach of adequate perception. Our ordinary intuitive representation of physical space is, as far as intuition goes, perfectly adequate; it does not need scientific amendment. But, on the other hand, science may find it convenient to adopt, for scientific purposes, considering maybe more refined ways of experiencing reality, different representations of physical space; our perceptual picture of space cannot stand in the way of this.[18] We must nonetheless refrain from interpreting what is only a scientific methodological strategy as a *correction* of perception.

I believe the distinctive aspect of Husserl’s treatment of the problems regarding spatial representation and the nature of geometry lies in his careful distinction among different conceptions of space, which were not always clearly separated by his contemporaries. Helmholtz, Riemann or Poincaré were mainly concerned with whether there is, from a *scientific* perspective, a *correct* geometry of space, and, if not, on what basis this or that spatial property can be attributed to physical space. They do not distinguish between represented space, physical space as our (far from passive) perceptual systems represent it, and the space of physical science, or even transcendent space. Husserl, on the contrary, carefully draws two distinctions: one, Kantian in spirit, between noumenal (or transcendent) and phenomenal (or represented) spaces; another, between pre-scientific and scientific representations of space. But, contrary to Kant, he thinks phenomenal space is dependent on reality in at least two ways: our space constituting functions were selected in an adaptive process, but, even so, they cannot fulfill their task to completion without some contribution from experience. Space-constituting functions are, precisely, *functions*, whose arguments the outer senses must provide. Remaining our senses and space-constituting functions what they are, Husserl thinks, *only one* output, *only one* space representation is possible, that which Euclidean geometry approximately describes.

This however does not mean that science, which is not only an affair of perception, but reason and convenience, cannot choose any other representation of space deemed more convenient. In fact, by distinguishing between the perceptual and the scientific representations of physical space Husserl opens the door to spatial representations in science that *are not, and need not be* that of perceptual intuition (later in life he came to believe that we may have to pay a high price for that, the dehumanization of science).

To conclude, it may be illuminating, I think, to compare my reading of Husserl with Weyl’s views on space representation, for, as already mentioned, the disciple was intellectually very close to his master.

The first chapter of *Space, Time, Matter* begins with a phenomenological analysis of our intuitive representation of physical space that, albeit succinct, renders Weyl’s

[18] I have Bergson’s critique of Einstein’s conception of time in mind.

views very clearly. For him, intuitive space is a *form* of phenomena, which is *perceived* as a continuous extended manifold. Due to the formal character of space the *same* content can, without changing in any way other than space position, occupy *any* spatial location; i.e. intuitive space is *homogeneous* (the same everywhere, and supposedly also *isotropic*, the same in any direction): "a space that can serve as a 'form of phenomena' is *necessarily* homogeneous [*my emphasis*]" (Weyl 1963, p. 86). Spatial regions are said to be *congruent* if they can in principle be occupied by the same content. For Weyl, then, homogeneity of space is a priori (it follows necessarily from space being a form) and the notion of congruence is intuitively given (in accordance with Husserl). But for Husserl, remember, perceptual intuition is a psychological, not a merely physical experience; congruence is not simply *given* in *passive* spatial experience. It remains then to see whether, for Weyl also, intuition, in particular the intuition of spatial form is a passive experience, or, contrarily, whether it involves actively the perceiving subject.

In his *Philosophy of Mathematics and Natural Science*, particularly §18, entitled "The Problem of Space", Weyl considers the intuitive representation of space firstly from the point of view of the physiology of vision, borrowing heavily from Helmholtz's *Physiological Optics*, among others. It is obvious from his considerations that Weyl can appreciate the difference between mere sense perception and intuition, the physical and psychological experiences Husserl mentions. A quote he approvingly takes from Fichte makes this clear: "I am originally not only sentient but also intuiting" (Weyl 1963, p. 127). In this same work he also says, agreeing explicitly with Husserl, that "the data of sensation are animated by 'interpretations', and only in union with them do they perform the 'function of representation'" (p. 119). In short, perceiving is an active act of constitution.

Weyl also agrees with Husserl on the necessity that goes with the intuition of space. He tells us so by quoting from Husserl (1931), §150, on the distinctive aspects of visual experience: "All these facts, allegedly mere contingencies of spatial intuitions that are alien to the 'true', 'objective' space, reveals themselves, except for minor empirical particularities, as essential necessities" (Weyl 1963, p. 129). For Weyl, as for Husserl, we cannot but represent space *intuitively* the way we do, except for minor, purely empirical details (like the three dimensions of space, which Weyl and Husserl take as a contingent fact—*if* experience had so suggested, our space-representing functions *would have represented* space with a different number of dimensions).

One of the forms of the a priori, Weyl says, consists in "non-empirical laws (*Wesensgesetze*) [*laws of essence, JJS*] [that] express the manner in which data and strata of consciousness are founded upon each other, but do not claim to involve statements of fact; this line of pursuit culminated in Husserl's phenomenology" (Weyl 1963, p.134). For Husserl, as Weyl approvingly explains, the a priori is rooted on essential laws. Since the intuitive representation of physical space involves, for Weyl as well as for Husserl, immutable laws of constitution, the features of intuitive space are largely "except for minor empirical details", a priori (for Weyl these details have to do mainly with the particular value of the curvature of space). These laws, however, although immutable, are not indifferent to spatial experience, having been generated as adaptive *responses* to it. Says Weyl: "[…] the manner in which this intuition as an integrating factor penetrates the sense data and utilizes their material is largely conditioned by experience" (Weyl 1963, p. 130).

From the intuitively given notion of congruence Weyl defines that of rigidity: a *rigid body* is one that always occupies congruent regions of space. With the notions of congruence and congruent transformation (which captures conceptually the intuitive notion of motions of rigid bodies in space) he defines the concept of *straight line* (the line determined by two distinct points A and B is the set of all points of space that are transformed into themselves by all congruent transformations that take A and B into themselves). Given the notion of straight line, the concept of *betwenness* can easily be defined: a point A of a line divides it in two rays, if B and C are in different rays, A is said to lie between B and C.

Summarizing: for Weyl, geometrical concepts, particularly the fundamental ones of congruence and congruent transformation, are grounded on intuition. Intuitive space is a connected unbounded homogeneous isotropic continuous enveloping manifold. Geometry develops out of our effort to "capture" intuitive space conceptually and, in the limit, symbolically; the mathematical notion of transformation (resp. group of transformations) offers itself naturally as the conceptual correspondent of the intuitive notion of motion (resp. system of motions). With these concepts different geometrical systems can be introduced and eventually axiomatized. The three dimensions of intuitive space ("the most fundamental property of space is that its points form a tri-dimensional manifold", Weyl 1963, p. 84) are accidental, they neither belong analytically to the concept of space, nor are a priori in the sense given above. So, from a strictly mathematical perspective, the notion of space can be generalized to that of an n-dimensional manifold (of which the generic Riemannian manifold is a specification).

The structure of intuitive space, Weyl says, *necessarily* satisfies Euclidean geometry (cf. Weyl 1963, p. 135). For him, "there is no doubt that the conviction which Euclidean geometry carries for us is essentially due to our familiarity with the handling of that sort of bodies which we call rigid and of which it can be said that they remain the same under varying conditions" (Weyl 1963, p.78). In sharp contrast with the scientific representation of physical space, we do not have choices on how to represent space *intuitively*. "This view", he continues, "does not contradict physics, in so far as physics adheres to the Euclidean quality of the infinitely small neighborhood of a point O (at which the ego happens to be at the moment)" (Weyl 1963, p. 135). I.e. physics is free to choose no matter which geometry for physical space provided it is locally Euclidean. The gap between physical and intuitive spaces grows as one distances from the "center" where the ego is (carrying with him his Euclidean intuitive space).

Weyl's main interest, however, does not lie on the genesis of our intuitive representation of space, but on the constitution of our *scientific* picture of physical space ("intuitive space and intuitive time are […] hardly the adequate medium in which physics is to construct the external world", Weyl 1963, p. 113). He considers in particular whether our representation of physical space can be put to empirical test. Answering Helmholtz's claim that any such attempt will always involve physical statements about the behavior of rigid bodies and light rays, Weyl says that a constructive theory can only be put to the test *as a whole*. And if so, we are allowed to suppose, science can adopt any geometrical theory of space making the whole geometry + physics adequate vis-à-vis the available empirical data. This immediately leads to a threefold partition of the concept of space: intuitive space, which has to conform only to immediate sensorial perception; physical space ("the ordering scheme of the real things, which enters as an integral component into the theoretical construction of the world", Weyl 1963, p. 134), which must represent a compromise between intuitive space and physical theories; and more or less arbitrarily conceived mathematical spaces.

Even a cursory reading of Weyl shows the extent to which he is in agreement with the views I attributed here to Husserl: (1) There are many conceptions of space: intuitive space, physical space (the space of physical sciences) and mathematical spaces. There is a genetic sequence of spatial representations, beginning with the intuitive and ending with the purely symbolic representation ("all knowledge, while it starts with intuitive description, tends towards symbolic reconstruction", Weyl 1963, p. 75). (2) Intuitive space is not simply abstracted from "unprocessed" spatial experience; it is on the contrary actively constituted by the subject from data of spatial perception according to immutable laws; there is a difference between passive perception and active intuition of space. (3) Intuitive space is necessarily Euclidean; we are not free to *perceive* space in any other guise. (4) But, for scientific purpose, we are free to *conceive* space in the way we deem best, provided the scientific representation of space is locally Euclidean—the geometrical structure of physical space must reach a compromise between intuition and logical, methodological or theoretical requirements of science. (5) For Weyl, our representation of space, in order to have objective validity (for all, not only for me) must relinquish content and undergo a purely formal symbolic reconstruction; i.e. only the formal aspects of space, which are the only that can be symbolically expressed, are truly objective. We have seen that Husserl also accorded great importance to symbolic languages as means of expressing fundamental geometrical truths and erect axiomatic systems where derivations must rely on symbolic manipulations only. But, more importantly, to the extent that they can be linguistically expressed at all, geometrical assertions can only express abstract formal properties of spatial relations (such as congruence, order and betwenness). Geometry, whether in axiomatic form or not, despite being given a definite realm—idealized perceptual space—can only convey formal truth. We can intuit geometric truths indirectly via perceptual intuitions (diagrams and other perceptual structures) because the latter *display*, in a materially filled context, the *same* formal content the former only *express* linguistically or symbolically. Geometric intuition requires that formal aspects of perceptual structures be recognized as *identical* to those that are only emptily represented in geometrical assertions (this is how, I think, we can interpret Husserl's claim that perceptual structures can stand as visible symbols for geometric ones—formal identity providing the basis for the representational relation). In short, geometry, and knowledge in general, involve an interplay between subjective intuitions and objective symbolization. (6) Both Weyl and Husserl recognize the ideality of geometry. For the former: "the geometrical statements […] are merely ideal determinations, which taken in individual isolation lack any meaning verifiable by what is given. Only here and there does the entire network of ideal determinations touch upon experienced reality, and at these points it must ‘check’. That, expressed in the most general terms, may well be called *the geometrical method*” (Weyl 1963, p. 132).

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